Sharing Inspiration 2019 : The Power of Realization

Walk of the drunk Rover

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Problem statement

Walking drunkhard

A drunkard walks on the pavement in a straight line to return home. Completely drunk, he does not control his progress and he could as easily take a step forward as a step backward.

Problem statement

Questions ?

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- Will he return (one day ?) to his starting point ?

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- What is the probability that he gets there in n steps ?

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- Will he return (one day ?) to his starting point ?
- What is the probability that he gets there in n steps ?
- What is the mathematical expectation (expected value) of the number of steps required ?

Backward Forward

Coding

```
Define bf(n,d)=
Prgm
Send "CONNECT RV"
For i,1,n
Send "RV FORWARD eval(d)"
Send "RV BACKWARD eval(d)"
EndFor
EndPrgm
```

Random Backward Forward

Coding

```
Define random_bf()=
Prgm
Send "CONNECT RV"
© d is a random number ∈ {−1, 1}
d :=2*randInt(0,1)-1
If d=1 Then
Send "RV FORWARD 1"
Else
Send "RV BACKWARD 1"
EndIf
EndPrgm
```

Return to the starting point

Coding - First step

```
Define drunkard()=
Prgm
© n count the steps, x is the drunkard position
n :=0
x :=0
Send "CONNECT RV"
random_bf()
x :=x+d
n :=n+1
```

Return to the starting point

Coding - Next steps

While $x \neq 0$

random_bf()

x :=x+d

n :=n+1

EndWhile

Disp "steps",n

EndPrgm

Where did he go ?

Coding - First step

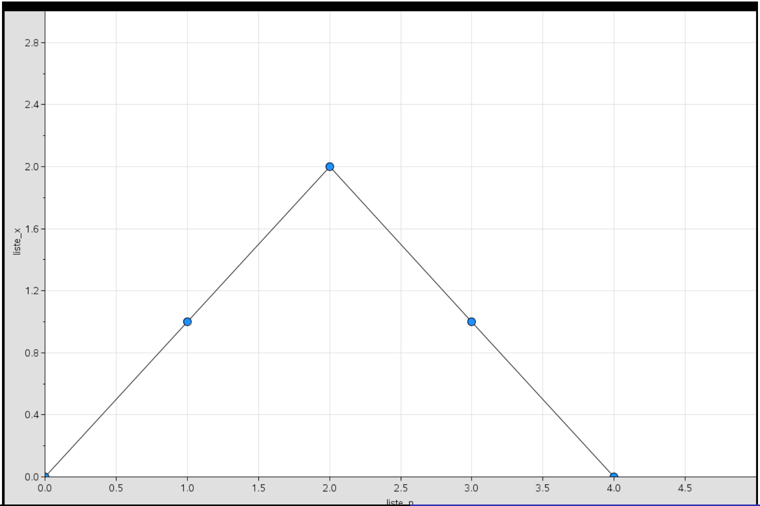
```
Define drunkard_pos()=
Prgm
n :=0 : x :=0
liste_n := {0} : liste_x := {0}
Send "CONNECT RV"
random_bf()
x :=x+d
n :=n+1
liste_x :=augment(liste_x,{x})
liste_n :=augment(liste_n,{n})
```

Where did he go ?

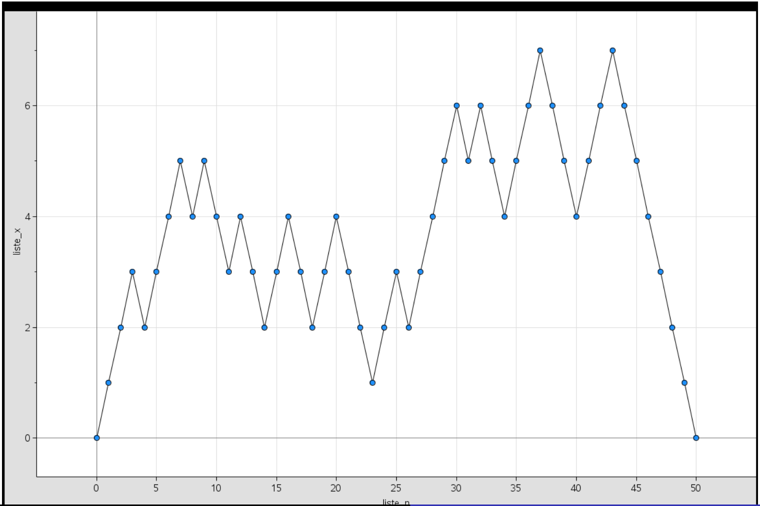
Coding - Next steps

```
While x ≠ 0
random_bf()
x :=x+d
n :=n+1
liste_x :=augment(liste_x,{x})
liste_n :=augment(liste_n,{n})
EndWhile
Disp "liste n=",liste_n
Disp "liste x=",liste_x
EndPrgm
```

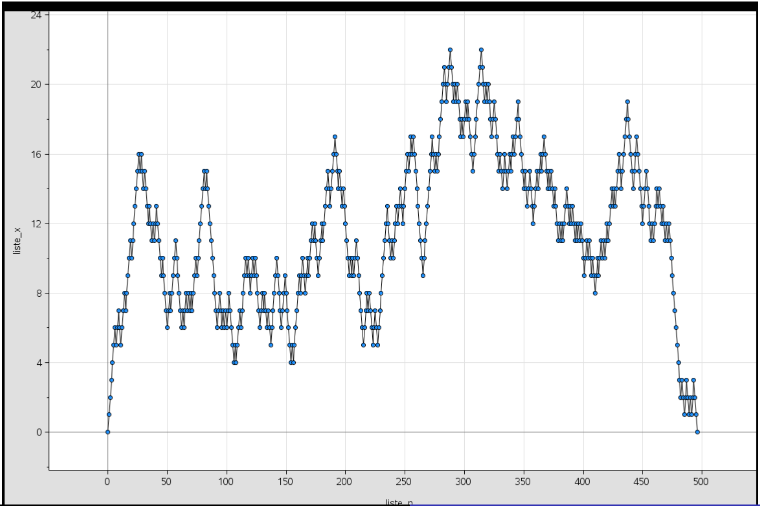
Viewing of a possible path



Viewing of a possible path



Viewing of a possible path



Question 1

Question 1 : Will he return ?

Does the drunkard always return to his starting position ?

If so, how many steps did he take ?

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Before asking questions about probabilities, let's start with statistical investigations.

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Does the drunkard always return to his starting position ?

If so, how many steps did he take ?

Before asking questions about probabilities, let's start with statistical investigations.

So let's repeat the "drunkard_far" program many times. This time, we neglect to roll the Rover.

Multiple random walk

Coding

```
Define drunkard_multi(trials)=
Prgm
steps :={}
distance :={}
For i,1,trials
drunkard()
steps :=augment(steps,{n})
distance :=augment(distance,{far})
EndFor
EndPrgm
```

Answer 1 : He's coming back !

Experimental conclusion

Experimentally, the drunkard always comes back to his starting point, even if sometimes he has to walk a lot !

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Experimental conclusion

Experimentally, the drunkard always comes back to his starting point, even if sometimes he has to walk a lot !

Question 2

Question 2

Can we calculate the probability that the drunkard will return to his starting point after n steps ?

First approach

Study of possible progression

The drunkhard walks n steps.

What is the probability that he will arrive at a distance x from the origin ?

First approach : Two methods

- Let's count the number of possible paths to reach x in n steps.

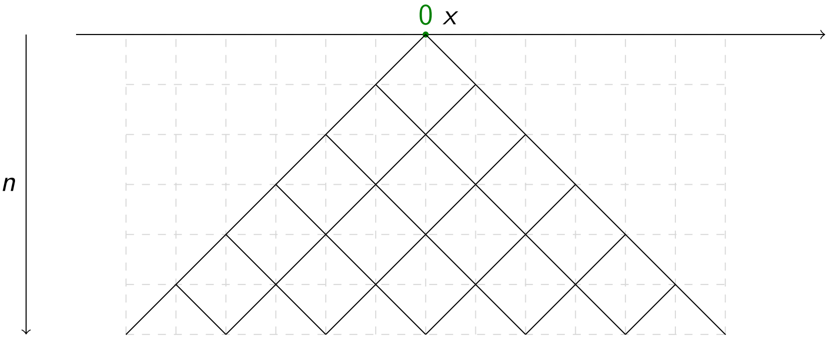
First approach : Two methods

- Let's count the number of possible paths to reach x in n steps.
- Let's use the binomial distribution

Counting number of paths

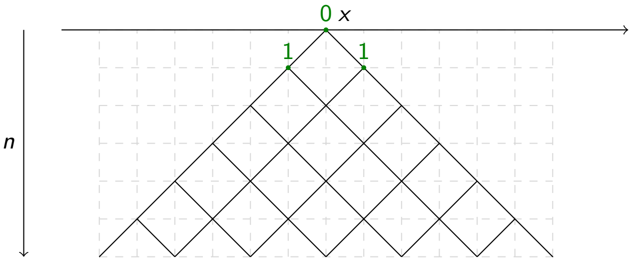
Let's show the drunkard's movements on a graph.

At the beginning ($n = 0$), he is at $x = 0$



Counting number of paths

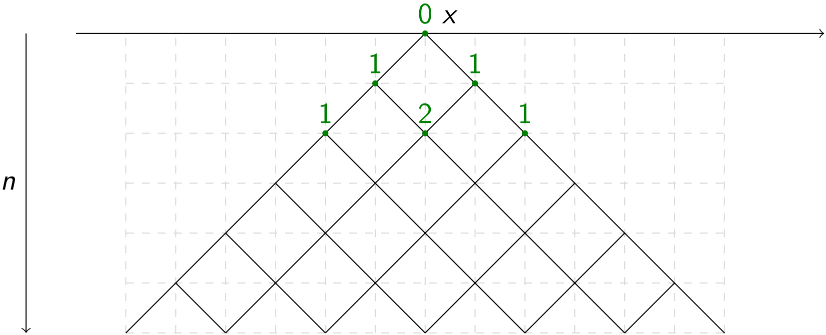
At the first step ($n = 1$), he can be either a step forward ($x = 1$) or a step backward $x = -1$



The green numbers shows the number of possible paths to reach the x position in n steps.

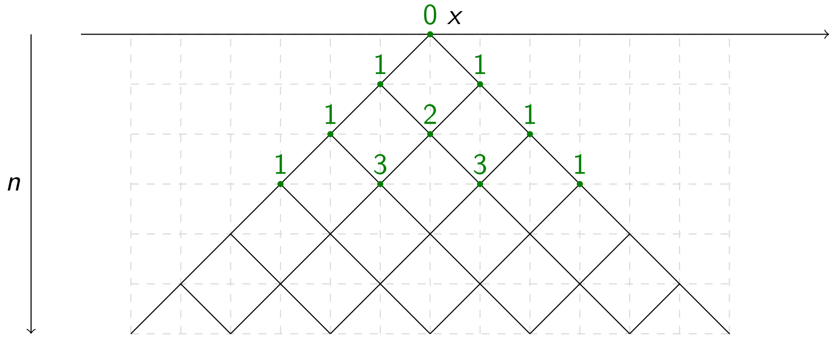
Counting number of paths

At the second step ($n = 2$), he can be at $x = -2$, $x = 0$ or $x = 2$



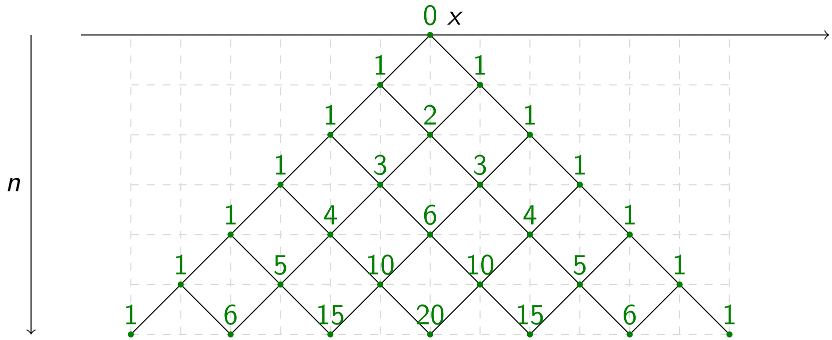
Counting number of paths

At the third step ($n = 3$), he can be at each abscissa $x = -3$, $x = -1$, $x = 1$ ou $x = 3$



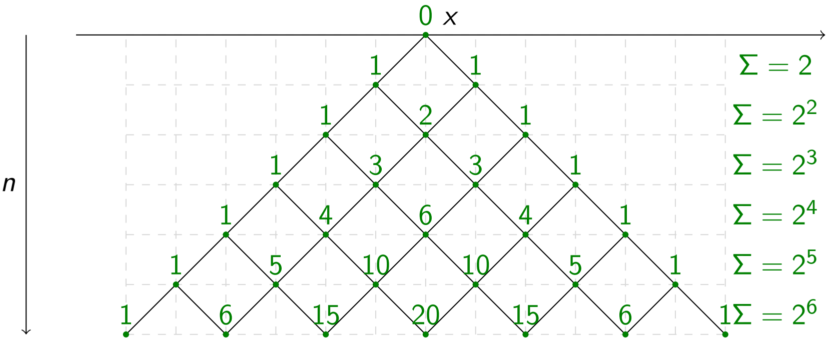
Counting number of paths

At n^{th} step, we get the following table (Pascal's triangle)



Counting number of paths

At n^{th} step, we get the following table (Pascal's triangle)

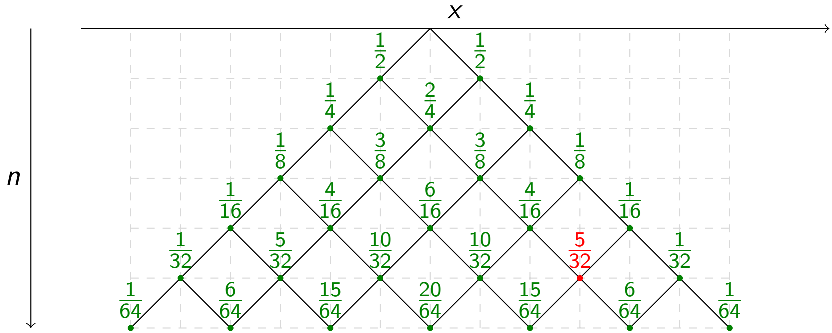


Probability computation

The probability $P(x,n)$ that the drunkard has reached the point $A(x,n)$ of abscissa x after n steps is calculated by the quotient of the number of paths arriving at A by the number of possible paths in n steps ($=2^n$).

Probability computation

$P(x, n)$



Theoretical way

In a theoretical way, to arrive at point $A(x, n)$, we must choose the position of the p $(+1)$ and the q (-1) in a such way as

$$\begin{cases} p+q=n \\ p-q=x \end{cases} \iff \begin{cases} p=\frac{n+x}{2} \\ q=\frac{n-x}{2} \end{cases}$$

Theoretical way

In a theoretical way, to arrive at point $A(x, n)$, we must choose the position of the p $(+1)$ and the q (-1) in a such way as

$$\begin{cases} p+q=n \\ p-q=x \end{cases} \Longleftrightarrow \begin{cases} p=\frac{n+x}{2} \\ q=\frac{n-x}{2} \end{cases}$$

and thus

$$P(x,n)=\frac{C_n^{\frac{n+x}{2}}}{2^n}$$

Binomial distribution

Using binomial distribution (forward = success, backward = failure) where X is the random variable representing the number of steps "forward"

$$P(x, n) = P(X = p)$$

Binomial distribution

Using binomial distribution (forward = success, backward = failure) where X is the random variable representing the number of steps "forward"

$$\begin{aligned} P(x, n) &= P(X = p) \\ &= P\left(X = \frac{x+n}{2}\right) \end{aligned}$$

Binomial distribution

Using binomial distribution (forward = success, backward = failure) where X is the random variable representing the number of steps "forward"

$$
\begin{aligned}
P(x,n) &= P(X = p) \\
&= P\left(X = \frac{x+n}{2}\right) \\
&= C_n^{\frac{n+x}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{n+x}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{n-x}{2}}
\end{aligned}
$$

Binomial distribution

Using binomial distribution (forward = success, backward = failure) where X is the random variable representing the number of steps "forward"

$$
\begin{aligned}
P(x,n) &= P(X = p) \\
&= P\left(X = \frac{x+n}{2}\right) \\
&= C_n^{\frac{n+x}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{n+x}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{n-x}{2}} \\
&= \frac{C_n^{\frac{n+x}{2}}}{2^n}
\end{aligned}
$$

First return to the starting point

Back to question 2

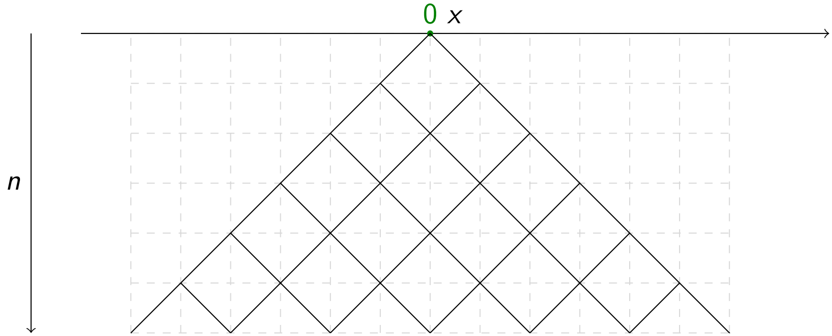
What is the probability that it comes back for the first time at the origin after n steps ?

First return to the starting point

To return to 0, the number of steps must be an even number $n = 2k$. The number of paths can be counted on a truncated triangle such as :

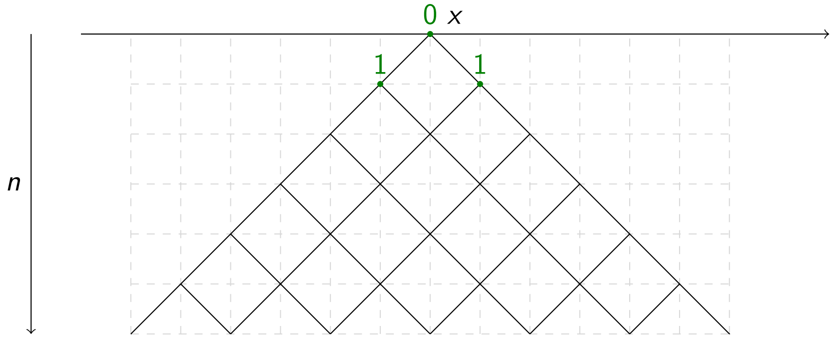
Counting

At the beginning ($n = 0$), the drunkhard is at $x = 0$



Counting

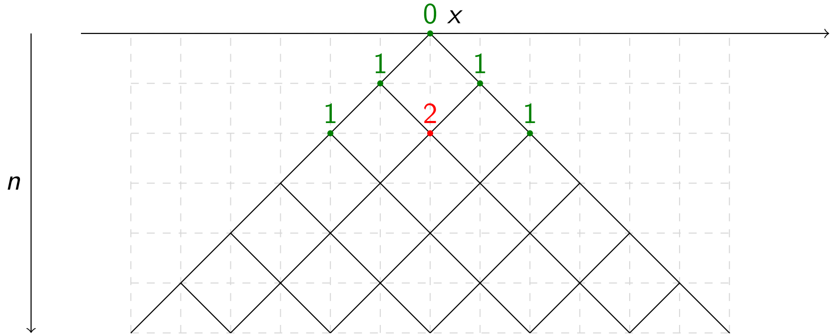
At the first step ($n = 1$),



The green numbers shows the number of possible paths to reach the x position in n steps.

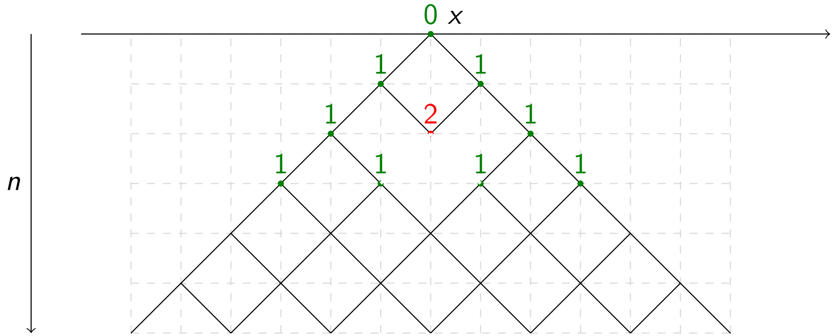
Counting

If, on the 2nd step ($n = 2$), the drunkard returns to $x = 0$, he stops.



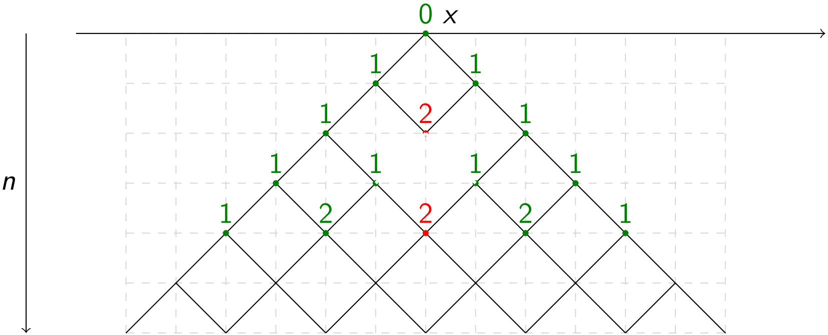
Counting

Otherwise, on the 3rd step ($n = 3$), the drunkard can be at abscissa $x = -3$ or $x = -1$ or $x = 1$ or $x = 3$



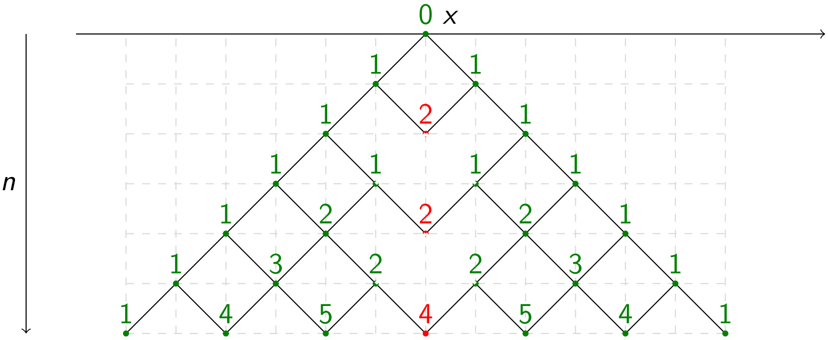
Counting

If, on the 4th step ($n = 2$), the drunkard returns to $x = 0$, he stops.



Counting

On the $6th$ step,we get the following table :



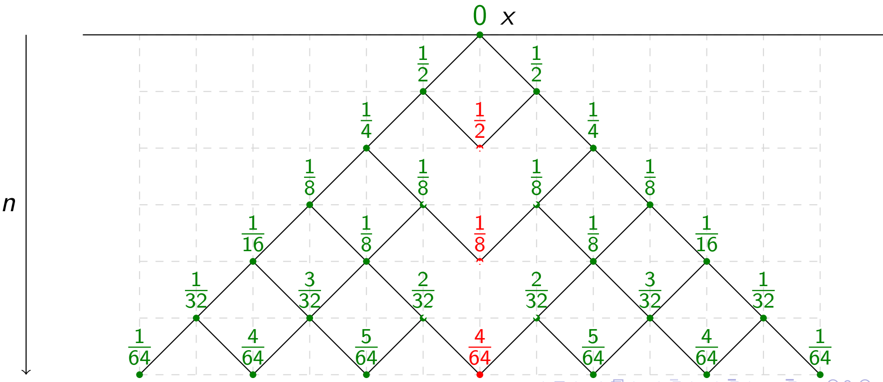
Probability

The probability $F(0, n)$ that the drunkhard arrives for the first time at $x = 0$ after n steps is the quotient of the number of paths arriving by the number of possible paths in n step $(= 2^n)$

- $F(0,2) = \frac{2}{2^2} = \frac{1}{2}$
- $F(0,4) = \frac{2}{2^4} = \frac{1}{8}$
- $F(0,6) = \frac{4}{2^6} = \frac{1}{16}$

Probability

$F(0, n) =$



Probability of the fisrt return in $2k$ steps

If $F(0,2k)$ is the probability of returning to the starting point after $n = 2k$ steps, we have

$$F(0,2k) = P(0,2k-2) - P(0,2k)$$

and we can prove that

$$F(0,2k) = \frac{P(0,2k)}{2k-1}$$

$P(0,2k)$ being the probability that the drunkard arrives at $A(0,2k)$ of abscissa 0 after $2k$ steps ie $P(0,2k) = \frac{C_{2k}^{k}}{2^{2k}}$

Proof

$$\begin{aligned}
P(0,2k-2)-P(0,2k) &= \frac{C_{2k-2}^{k-1}}{2^{2k-2}} - \frac{C_{2k}^{k}}{2^{2k}} \\
&= \frac{1}{2^{2k-2}}\left(\frac{(2k-2)!}{((k-1)!)^2} - \frac{(2k)!}{4(k!)^2}\right) \\
&= \frac{(2k-2)!}{2^{2k-2}}\left(\frac{4k^2-(2k-1)2k}{4(k!)^2}\right) \\
&= \frac{(2k-2)!}{2^{2k}}\left(\frac{2k}{(k!)^2}\right)\frac{2k-1}{2k-1} \\
&= \frac{(2k)!}{2^{2k}}\frac{1}{(k!)^2(2k-1)} \\
&= \frac{C_{2k}^{k}}{2^{2k}}\left(\frac{1}{(2k-1)}\right) \\
&= \frac{P(0,2k)}{(2k-1)}
\end{aligned}$$

Answer 2

Answer 2

Probability of first return

$$F(0,2k) = \frac{P(0,2k)}{(2k-1)} = \frac{C_{2k}^{k}}{(2k-1)\cdot 2^{2k}}$$

Question 3

Question 3

Can we calculate the expectation of steps required ?

Distribution

The distibution is therefore

k	1	2	3	4	5	$\dots$	k
$X_i = n$	2	4	6	8	10	$\dots$	$2k$
p_i	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{5}{128}$	$\frac{7}{256}$	$\dots$	$\frac{C_{2k}^{k}}{(2k-1) \cdot 2^{2k}}$

Expectation

Mathematical expectation of the number of steps to return at the origin for the first time.

$$E = P(0,2)\cdot 2 + P(0,4)\cdot 4 + \dots$$
$$= \sum_{k=0}^{+\infty} \frac{2k}{2k-1}\cdot\frac{(2k!)}{(k!)^2\cdot 2^{2k}}$$

But

$$\lim_{k\to+\infty} \underbrace{\frac{2k}{2k-1}}_{\to 1}\cdot\frac{(2k!)}{(k!)^2\cdot 2^{2k}} = \lim_{k\to+\infty} \frac{(k+1)}{4\cdot 1}\cdot\frac{k+2}{4\cdot 2}\dots\frac{2k}{4\cdot k} \neq 0$$

Thus the series is divergent.

Answer 3

Answer 3

The expectation of steps required doesn't exist.

Answer 3

Answer 3

The expectation of steps required doesn't exist.

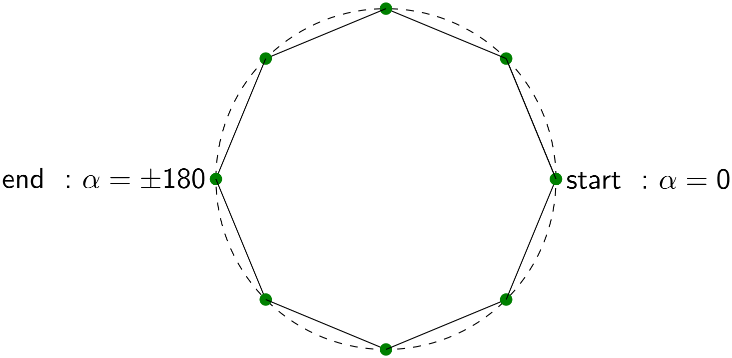
Walking on a polygon

Problem statement

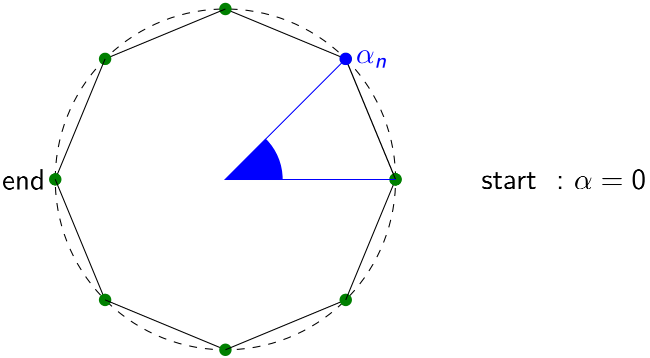
The drunkard moves randomly walking through the vertices of a regular polygon At each step, he goes from one vertex to one of the two adjacent vertices.

How many steps will it take to reach the vertex opposite the starting point ?

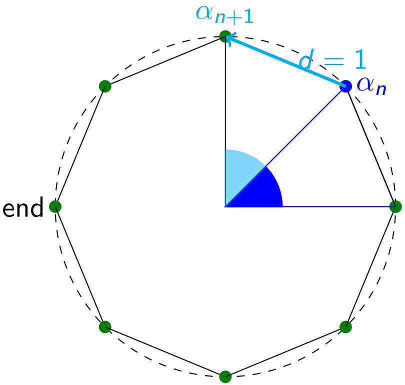
Principle



Walking on a polygon : Principle

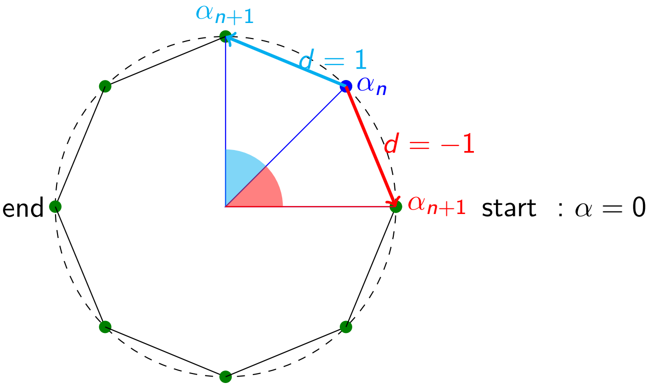


Walking on a polygon : Principle



start : $\alpha = 0$

Walking on a polygon : Principle



Walking on a polygon

Coding

Define polygon_walk(r,nside)=

Prgm

Send "CONNECT RV"

$\alpha := 0$: n :=0

Send "RV TO POLAR eval(r) eval(alpha)"

While $\alpha \neq 180$ and $\alpha \neq -180$

d :=2*randInt(0,1)-1

n :=n+1

$\alpha := \alpha + \dfrac{d * 360}{nside}$

Send "RV TO POLAR eval(r) eval(alpha)"

EndWhile

Disp "steps",n

Multiple walk on a polygon

Coding

```
Define poly_multi(trials)=
Prgm
steps :={}
For i,1,trials
polygon_walk(r,n)
steps :=augment(steps,{n})
EndFor
Disp pas
EndPrgm
```

Reach the opposite vertex ?

Question

What is the average number of steps to reach the opposite vertex ?

- square

Reach the opposite vertex ?

Question

What is the average number of steps to reach the opposite vertex ?

- square
- hexagon

Reach the opposite vertex ?

Question

What is the average number of steps to reach the opposite vertex ?

- square
- hexagon
- others polygon

Simulations on a square

Coding

```
Define poly_multi(trials)=
Prgm
steps :={}
For i,1,trials
polygon_walk(3,4)
steps :=augment(steps,{n})
EndFor
Disp pas
EndPrgm
```

Statistical results

	A nbr_pas	B	C	D	E	F	G	H	I	J	K	L
=	=pas				=OneVar('nbr_							
1	8			Titre	Statistiques ...							
2	2			$\bar{x}$	3.7							
3	2			Σx	74.							
4	2			Σx^2	372.							
5	4			sx := s_{n-}...	2.27342							
6	2			σx := σ_n...	2.21585							
7	2			n	20.							
8	4			MinX	2.							
9	2			Q_1X	2.							
10	2			MedianX...	3.							
11	2			Q_3X	4.							
12	6			MaxX	10.							
13	10			SSX := Σ..	98.2							
14	4											
15	4											
16	2											
17	2											
18	4											
19	6											
20	4											

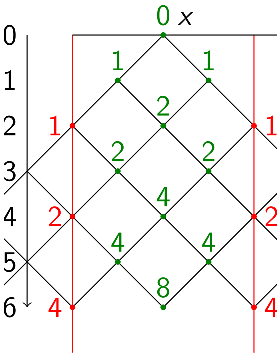
B1

Statistical results

	A nbr_pas	B	C	D	E	F	G	H	I	J	K	L
=	=pas				=OneVar('nbr_							
1	8			Titre	Statistiques ...							
2	2			$\bar{x}$	3.7							
3	2			Σx	74.							
4	2			Σx²	372.							
5	4			sx := sn-...	2.27342							
6	2			σx := σn...	2.21585							
7	2			n	20.							
8	4			MinX	2.							
9	2			Q_1X	2.							
10	2			MedianX...	3.							
11	2			Q_3X	4.							
12	6			MaxX	10.							
13	10			SSX := Σ..	98.2							
14	4											
15	4											
16	2											
17	2											
18	4											
19	6											
20	4											

Probability

On a truncated graph such as this one, the numbers are the numbers of path arriving to this point.



Probability

The number of steps must be even and greater than or equal to 2, $n = 2k + 2$.

We can prove by induction that the number of paths arriving to $A(2, 2k + 2)$ (on the right side) is equal to 2^k.

The number of paths arriving to $A'(2, 2k + 2)$ (on the left side) is also 2^k.

So, the number of paths going to the opposite vertex (by right ou by left) is

$$N^2_{2k+2} = 2 \cdot 2^k = 2^{k+1}$$

Distribution and expectation

The probability to reach the opposite vertex with n steps is :

$$P(2,n) = \frac{2 \cdot 2^k}{2^{2k+2}} = \frac{1}{2} \cdot \frac{1}{2^k}$$

The distribution is

k	0	1	2	3	$\ldots$	k
x	2	4	6	8	$\ldots$	$2k+2$
p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\ldots$	$\frac{1}{2^{k+1}}$

Thus $E = \frac{1}{2}\sum_{k=0}^{\infty}(2k+2) \cdot \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{k+1}{2^k} = ?$

Expectation - demonstration

$$\begin{aligned}
E &= \sum_{k=0}^{\infty} \frac{k+1}{2^k} \\
&= \sum_{k=0}^{\infty} \frac{k}{2^k} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k}}_{\to 2 (SG)} \\
&= \sum_{k=1}^{\infty} \frac{k}{2^k} + 2 \\
&= \sum_{k=0}^{\infty} \frac{k+1}{2^{k+1}} + 2 \\
&= \frac{1}{2} E + 2
\end{aligned}$$

Thus $E = \frac{1}{2} E + 2 \iff E = 4$

Walk on a square - Conclusion

Reach the opposite vertex in a square

The average number of steps to reach the opposite vertex of a square is 4.

Reach the opposite vertex in an hexagon

Reach the opposite vertex

What is the average number of steps to reach the opposite vertex of an hexagon ?

Simulations on a hexagon

Coding

```
Define poly_multi(trials)=
Prgm
steps :={}
For i,1,trials
polygon_walk(3,6)
steps :=augment(steps,n)
EndFor
Disp steps
EndPrgm
```

Statistical results

	A nbr_pas	B	C	D	E	F	G	H	I	J	K	L
=	=pas				=OneVar('nbr_							
1	11			Titre	Statistiques ...							
2	7			$\bar{x}$	8.4							
3	3			Σx	168.							
4	5			Σx^2	1980.							
5	11			sx := sn-...	5.47146							
6	9			σx := σn...	5.33292							
7	5			n	20.							
8	19			MinX	3.							
9	11			Q_1X	5.							
10	5			MedianX...	7.							
11	7			Q_3X	10.							
12	7			MaxX	25.							
13	9			SSX := Σ..	568.8							
14	25											
15	3											
16	3											
17	9											
18	9											
19	3											
20	7											

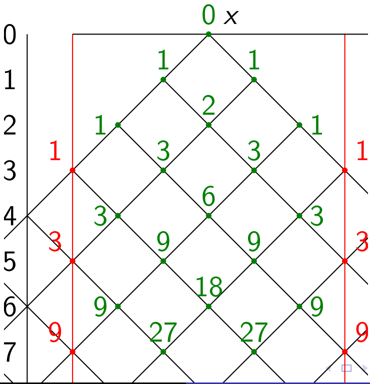
B1

Statistical results

	A nbr_pas	B	C	D	E	F	G	H	I	J	K	L
=	=pas				=OneVar('nbr_							
1	11			Titre	Statistiques ...							
2	7			$\bar{x}$	8.4							
3	3			Σx	168.							
4	5			Σx²	1980.							
5	11			sx := sn-...	5.47146							
6	9			σx := σn...	5.33292							
7	5			n	20.							
8	19			MinX	3.							
9	11			Q₁X	5.							
10	5			MedianX...	7.							
11	7			Q₃X	10.							
12	7			MaxX	25.							
13	9			SSX := Σ..	568.8							
14	25											
15	3											
16	3											
17	9											
18	9											
19	3											
20	7											

Probability

On a truncated graph such as this one, the numbers are the numbers of path arriving to this point.



Probability

The number of steps n to reach the opposite vertex must be an odd number and must be greater or equal to 3, $n = 2k + 3$.

We can prove by induction that the number of paths arriving to $A(3, 2k + 3)$ (on the right side), is 3^k.

So, the number of paths going to the opposite vertex (by right ou by left) is

$$N^3_{2k+3} = 2 \cdot 3^k$$

Distribution and expectation

The probability to reach the opposite vertex with n steps is

$$P(3,n) = \frac{2\cdot 3^k}{2^{2k+3}} = \frac{2\cdot 3^k}{8\cdot 2^{2k}} = \frac{1}{4}\cdot\frac{3^k}{4^k}$$

The distribution is

k	0	1	2	3	...	k
x	3	5	7	9	...	$2k+3$
p	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{27}{256}$	...	$\frac{1}{4}\cdot\frac{3^k}{4^k}$

Distribution and expectation

Thus

$$E = \frac{1}{4}\sum_{k=0}^{\infty}(2k+3)\frac{3^k}{4^k} = ?$$

Distribution and expectation

Thus

$$E = \frac{1}{4}\sum_{k=0}^{\infty}(2k+3)\frac{3^k}{4^k} = \frac{1}{2}\underbrace{\sum_{k=0}^{\infty}\frac{k\cdot 3^k}{4^k}}_{S_1} + \underbrace{\sum_{k=0}^{\infty}\frac{3^{k+1}}{4^{k+1}}}_{S_2}$$

Distribution and expectation

Thus

$$E = \frac{1}{4}\sum_{k=0}^{\infty}(2k+3)\frac{3^k}{4^k} = \frac{1}{2}\underbrace{\sum_{k=0}^{\infty}\frac{k\cdot 3^k}{4^k}}_{S_1} + \underbrace{\sum_{k=0}^{\infty}\frac{3^{k+1}}{4^{k+1}}}_{S_2}$$

S_2 : SG de $u_1 = \frac{3}{4}$ et $q = \frac{3}{4}$. Thus $S_2 = \dfrac{3}{4}\cdot\dfrac{1}{\frac{3}{4}} = 3$

Expectation - démonstration

$$
\begin{aligned}
S_1 &= \sum_{k=0}^{\infty} k \cdot \frac{3^k}{4^k} \\
&= \sum_{k=1}^{\infty} k \cdot \frac{3^k}{4^k} \\
&= \sum_{k=0}^{\infty} (k+1) \cdot \frac{3^{k+1}}{4^{k+1}} \\
&= \sum_{k=0}^{\infty} k \cdot \frac{3^{k+1}}{4^{k+1}} + \sum_{k=0}^{\infty} \frac{3^{k+1}}{4^{k+1}} \\
&= \frac{3}{4} S_1 + S_2
\end{aligned}
$$

Thus $S_1 = \frac{3}{4} S_1 + 3 \iff S_1 = 12$ et $E = 6 + 3 = 9$

Walk on an hexagon : Conclusion

Reach the opposite vertex in an hexagon

The average number of steps to reach the opposite vertex of an hexagon is 9.

Octagon - Monte Carlo method

	A nbr_pas	D	E
=	=pas		=OneVar('nbr_
1	8	Titre	Statistiques ...
2	8	$\bar{x}$	16.12
3	28	Σx	1612.
4	6	Σx²	41560.
5	6	sx := sn-...	12.5427
6	10	σx := σn...	12.4798
7	24	n	100.
8	22	MinX	4.
9	8	Q_1X	8.
10	20	MedianX...	12.
11	10	Q_3X	21.
12	66	MaxX	66.
13	36	SSX := Σ..	15574.6
14	6		

A1 =8

Octagon - Monte Carlo method

	A nbr_pas	B	C	D	E	F	G	H	I
=	=pas				=OneVar('nbr_				
1	8			Titre	Statistiques				
2	8			$\bar{x}$	16.12				
3	28			Σx	1612.				
4	6			Σx^2	41560.				
5	6			sx := sn-...	12.5427				
6	10			σx := σn...	12.4798				
7	24			n	100.				
8	22			MinX	4.				
9	8			Q_1X	8.				
10	20			MedianX...	12.				
11	10			Q_3X	21.				
12	66			MaxX	66.				
13	36			SSX := Σ..	15574.6				
14	6								

A1 =8

Decagon - Monte Carlo method

	A nbr_pas	B	C	D	E	F	G	H	I
=	=pas				=OneVar('nbr_				
1	63			Titre	Statistiques ...				
2	23			$\bar{x}$	24.62				
3	17			Σx	2462.				
4	15			Σx²	97500.				
5	13			sx := sn-...	19.3024				
6	55			σx := σn...	19.2056				
7	59			n	100.				
8	13			MinX	5.				
9	31			Q₁X	11.				
10	11			MedianX...	18.				
11	33			Q₃X	33.				
12	33			MaxX	119.				
13	13			SSX := Σ..	36885.6				
14	29								

A1 =63

Decagon - Monte Carlo method

	A nbr_pas	B	C	D	E	F	G	H	I
=	=pas				=OneVar('nbr_				
1	63			Titre	Statistiques				
2	23			x̄	24.62				
3	17			Σx	2462.				
4	15			Σx^2	97500.				
5	13			sx := sn-...	19.3024				
6	55			σx := σn...	19.2056				
7	59			n	100.				
8	13			MinX	5.				
9	31			Q_1X	11.				
10	11			MedianX...	18.				
11	33			Q_3X	33.				
12	33			MaxX	119.				
13	13			SSX := Σ..	36885.6				
14	29								

A1 =63

Conjecture

Expectation in a polygone

Conjecture - Answer

In a polygon with $2n$ sides, the average number of steps leading the drunkhard to the opposite vertex is equal to n^2.

Conjecture

Expectation in a polygone

Conjecture - Answer

In a polygon with $2n$ sides, the average number of steps leading the drunkhard to the opposite vertex is equal to n^2.

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